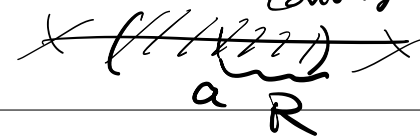
6.1 - Review of Power Series

A power series has the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

interval of convergence



In our work, $a = 0$.

So $$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$= c_0 + c_1 x + c_2 x^2 + \sum_{n=3}^{\infty} c_n x^n$$

Definition: A function is **analytic at a point a** if it can be represented with a power series centered at a with radius $R \neq 0$.

Example: Rewrite the given expression using a single power series whose general term involves x^k.

→ using one sigma

$$\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^{n+2}$$

Let $k = n-1$
Then $n = k+1$

- Make exponents match $$\sum_{k=0}^{\infty} (k+1) c_{k+1} x^k + \sum_{k=2}^{\infty} 3 c_{k-2} x^k$$

$k=0$ $\quad k=1$

- Make indices match $$= c_1 + 2c_2 x + \sum_{k=2}^{\infty} (k+1) c_{k+1} x^k + \sum_{k=2}^{\infty} 3 c_{k-2} x^k$$

$$= c_1 + 2c_2 x + \sum_{k=2}^{\infty} \left[(k+1) c_{k+1} + 3 c_{k-2}\right] x^k$$

Find a power series solution $y = \sum_{n=0}^{\infty} c_n x^n$ of the given linear first-order differential equation.

$4m+1=0 \Rightarrow m=-\frac{1}{4}$

$y = ce^{-1/4 x}$

$4y' + y = 0$

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \cdots$$

The starting value for the index increases when we differentiate

$$\sum_{n=1}^{\infty} 4n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

- Make exponents match (x^k)

$$\sum_{k=0}^{\infty} 4(k+1) c_{k+1} x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=0}^{\infty} \left[4(k+1) c_{k+1} + c_k\right] x^k = 0$$

$$4(k+1) c_{k+1} + c_k = 0$$ (all the terms are 0)

Solve for the constant that has the higher subscript.

Recurrence relation

$$c_{k+1} = -\frac{1}{4(k+1)} c_k, \quad k = 0, 1, 2, \ldots$$

$k=0 \qquad c_1 = -\frac{1}{4}c_0$

$k=1 \qquad c_2 = -\frac{1}{8}c_1 = -\frac{1}{8}\left(-\frac{1}{4}c_0\right) = \frac{1}{32}c_0$

$k=2 \qquad c_3 = -\frac{1}{12}c_2 = -\frac{1}{12}\left(\frac{1}{32}c_0\right) = -\frac{1}{384}c_0$

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots$$

Most of the time, you'll stop around here

Here, $y = c_0 - \frac{1}{4}c_0x + \frac{1}{32}c_0x^2 - \frac{1}{384}c_0x^3 + \cdots$

$$y = c_0\left(1 - \frac{1}{4}x + \frac{1}{2}\cdot\frac{1}{4^2}x^2 - \frac{1}{6}\cdot\frac{1}{4^3}x^3 + \cdots\right)$$

Let $c = c_0$

$$y = c\left[1 + \frac{-1}{4x} + \frac{1}{2}\left(-\frac{1}{4x}\right)^2 + \frac{1}{3!}\left(-\frac{1}{4x}\right)^3 + \cdots\right]$$

$$\boxed{y = c\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x}{4}\right)^n = ce^{-x/4}}$$

Example: The given function is analytic at $a = 0$. Use appropriate series and multiplication to find the first four nonzero terms of the Maclaurin series of the given function.

$e^{-x} \cos x$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^{-x} \cos x = \left(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right)$$

$$= 1 - x + \cancel{\tfrac{1}{2}x^2} - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^4 + \cdots$$

$$\cancel{-\tfrac{1}{2}x^2} + \tfrac{1}{2}x^3 - \tfrac{1}{4}x^4 + \cdots$$

$$+\tfrac{1}{24}x^4 - \tfrac{1}{24}x^5 + \tfrac{1}{48}x^6 + \cdots$$

$$-\tfrac{1}{720}x^6 + \cdots$$

$$e^{-x} \cos x = 1 - x + \tfrac{1}{3}x^3 - \tfrac{1}{6}x^4 + \cdots$$